Optimal Control and Planning

CS 294-112: Deep Reinforcement Learning

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Class Notes

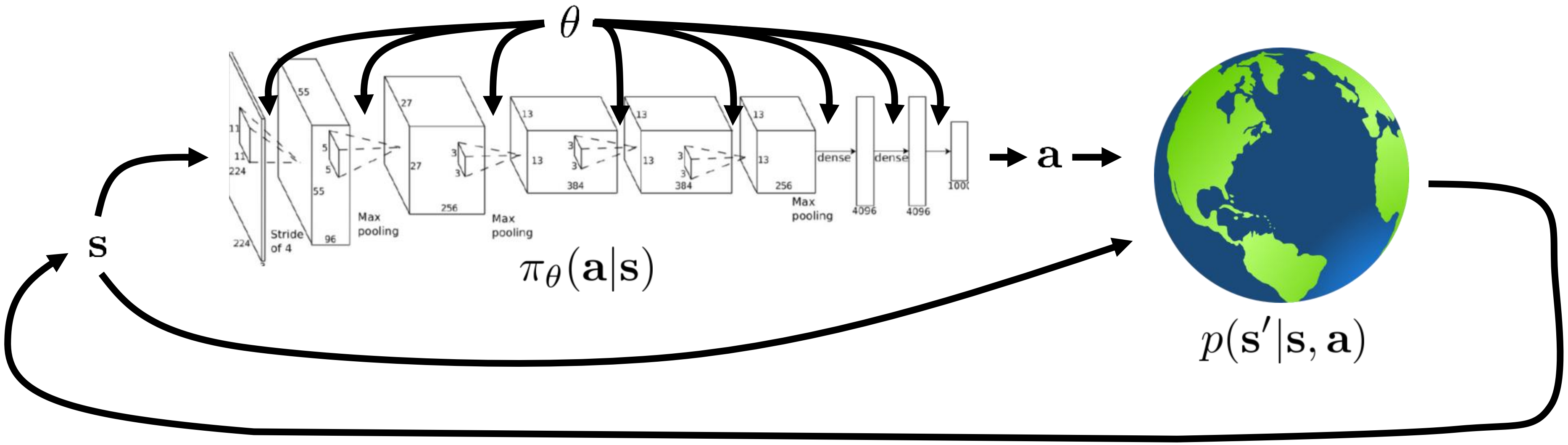
1. Homework 2 is due today, at 11:59 pm
2. Homework 3 comes out tonight
   - Start early, this one will take a bit longer!

Today's Lecture

1. Introduction to model-based reinforcement learning
2. What if we know the dynamics? How can we make decisions?
3. Stochastic optimization methods
4. Monte Carlo tree search (MCTS)
5. Trajectory optimization

- Goals:
  - Understand how we can perform planning with known dynamics models in discrete and continuous spaces

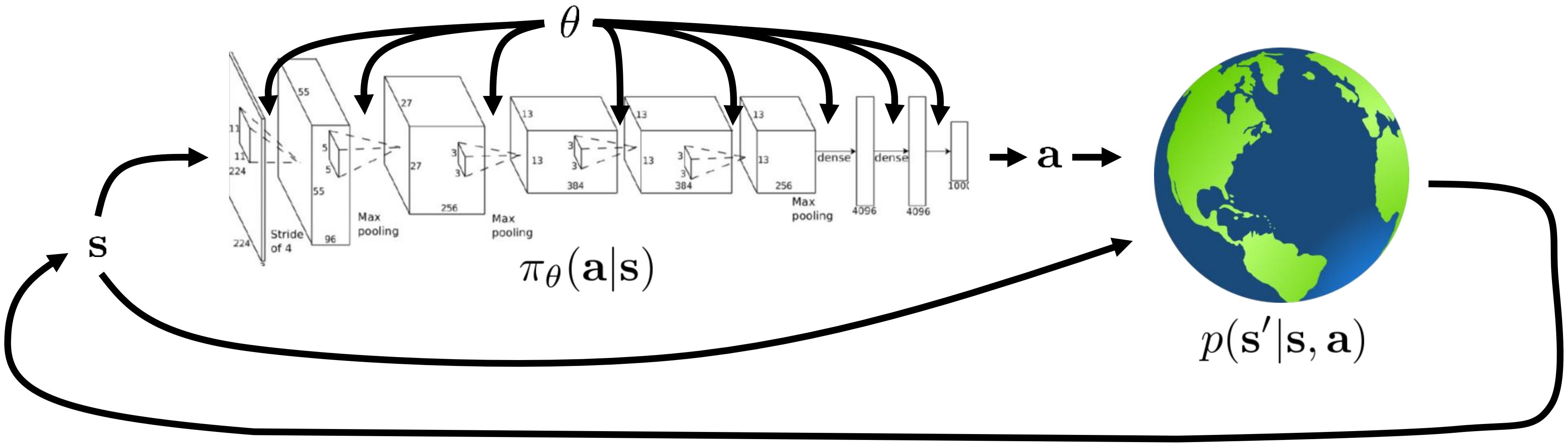
Recap: the reinforcement learning objective



$$\underbrace{p_\theta(\mathbf{s}_1,\mathbf{a}_1,\ldots,\mathbf{s}_T,\mathbf{a}_T)}_{\pi_\theta(\tau)} = p(\mathbf{s}_1)\prod_{t=1}^{T}\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)p(\mathbf{s}_{t+1}|\mathbf{s}_t,\mathbf{a}_t)$$

$$\theta^\star = \arg\max_\theta E_{\tau\sim p_\theta(\tau)}\left[\sum_t r(\mathbf{s}_t,\mathbf{a}_t)\right]$$

Recap: model-free reinforcement learning



$$\underbrace{p_\theta(\mathbf{s}_1,\mathbf{a}_1,\ldots,\mathbf{s}_T,\mathbf{a}_T)}_{\pi_\theta(\tau)} = p(\mathbf{s}_1)\prod_{t=1}^{T}\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)p(\mathbf{s}_{t+1}|\mathbf{s}_t,\mathbf{a}_t)$$

assume this is unknown

don't even attempt to learn it

$$\theta^\star = \arg\max_\theta E_{\tau\sim p_\theta(\tau)}\left[\sum_t r(\mathbf{s}_t,\mathbf{a}_t)\right]$$

What if we knew the transition dynamics?

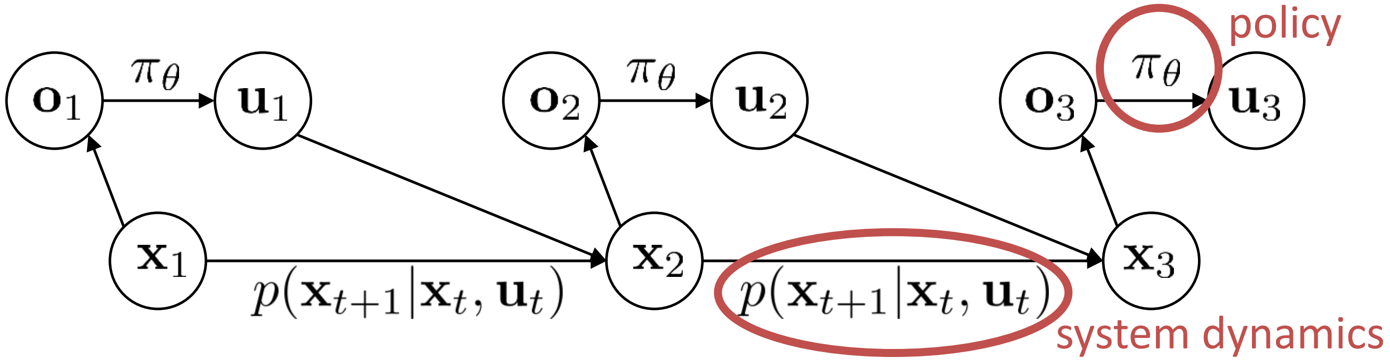
- Often we do know the dynamics
  1. Games (e.g., Go)
  2. Easily modeled systems (e.g., navigating a car)
  3. Simulated environments (e.g., simulated robots, video games)
- Often we can learn the dynamics
  1. System identification – fit unknown parameters of a known model
  2. Learning – fit a general-purpose model to observed transition data

Does knowing the dynamics make things easier?

Often, yes!

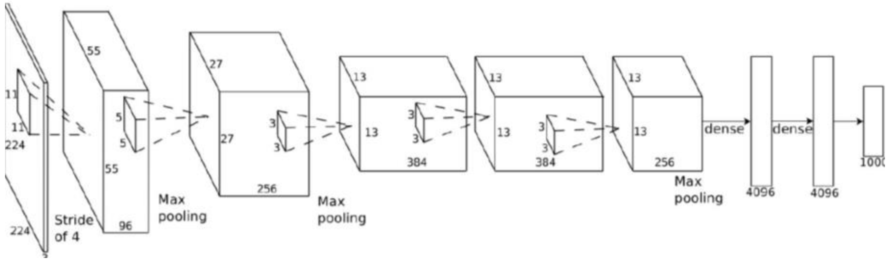
Model-based reinforcement learning

1. Model-based reinforcement learning: learn the transition dynamics, then figure out how to choose actions
2. Today: how can we make decisions if we *know* the dynamics?
   a. How can we choose actions under perfect knowledge of the system dynamics?
   b. Optimal control, trajectory optimization, planning
3. Next week: how can we learn *unknown* dynamics?
4. How can we then also learn policies? (*e.g. by imitating optimal control*)



The objective

$\mathbf{s}_t$



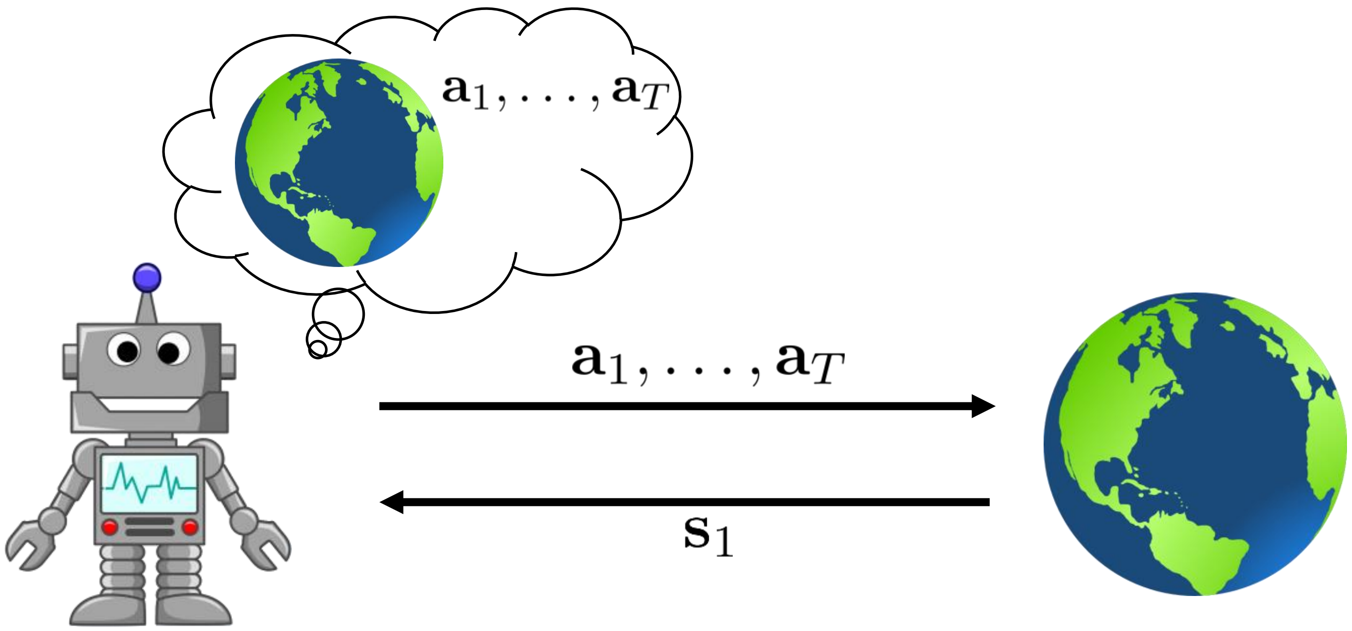
$\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)$

$\mathbf{a}_t$

$$\min_{\mathbf{a}_1,\ldots,\mathbf{a}_T} \log p(\text{eaten by tiger}|\mathbf{a}_1,\ldots,\mathbf{a}_T)$$

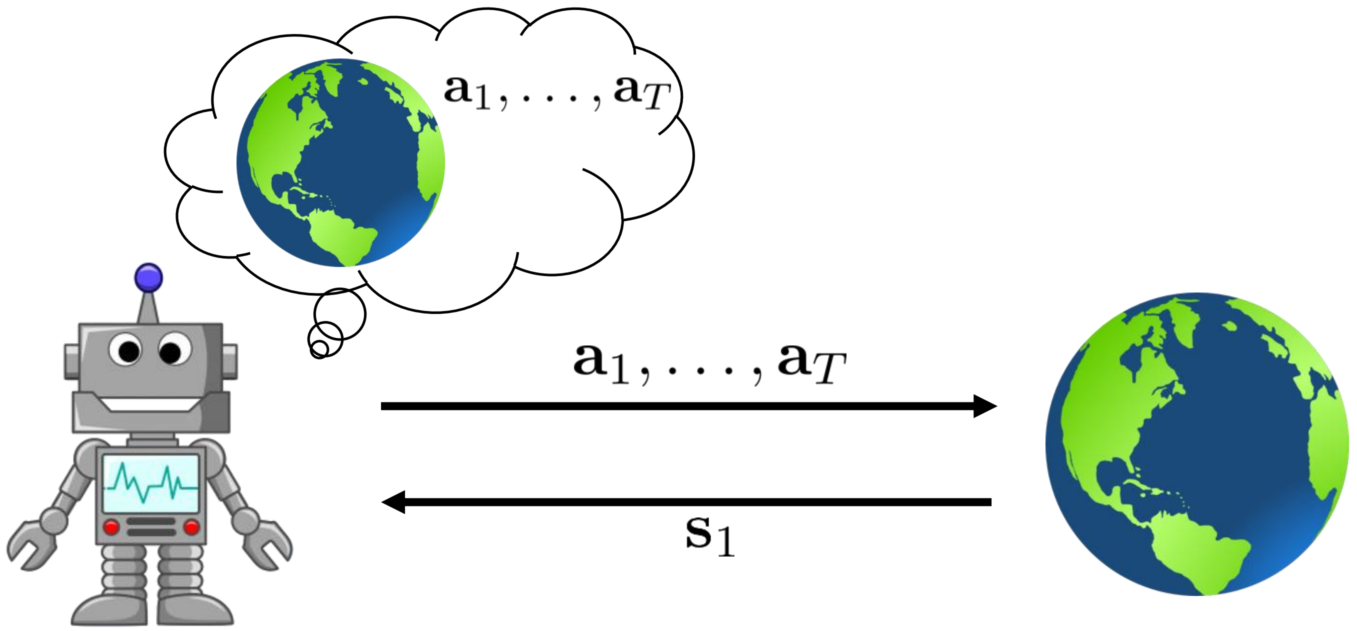
$$\min_{\mathbf{a}_1,\ldots,\mathbf{a}_T} \sum_{t=1}^{T} c(\mathbf{s}_t,\mathbf{a}_t) \text{ s.t. } \mathbf{s}_t = f(\mathbf{s}_{t-1},\mathbf{a}_{t-1})$$

The deterministic case



$$\mathbf{a}_1, \ldots, \mathbf{a}_T = \arg \max_{\mathbf{a}_1, \ldots, \mathbf{a}_T} \sum_{t=1}^{T} r(\mathbf{s}_t, \mathbf{a}_t) \text{ s.t. } \mathbf{a}_{t+1} = f(\mathbf{s}_t, \mathbf{a}_t)$$

The stochastic open-loop case



$$p_\theta(\mathbf{s}_1, \ldots, \mathbf{s}_T | \mathbf{a}_1, \ldots, \mathbf{a}_T) = p(\mathbf{s}_1) \prod_{t=1}^{T} p(\mathbf{s}_{t+1} | \mathbf{s}_t, \mathbf{a}_t)$$

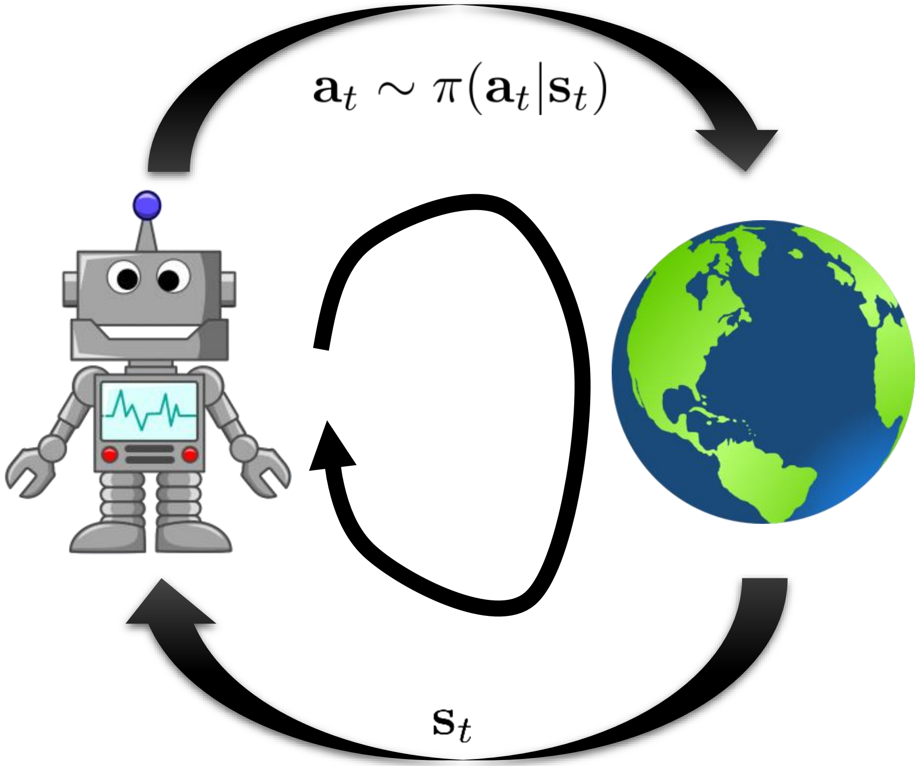
why is this suboptimal?

$$\mathbf{a}_1, \ldots, \mathbf{a}_T = \arg\max_{\mathbf{a}_1, \ldots, \mathbf{a}_T} E\left[\sum_t r(\mathbf{s}_t, \mathbf{a}_t) | \mathbf{a}_1, \ldots, \mathbf{a}_T\right]$$

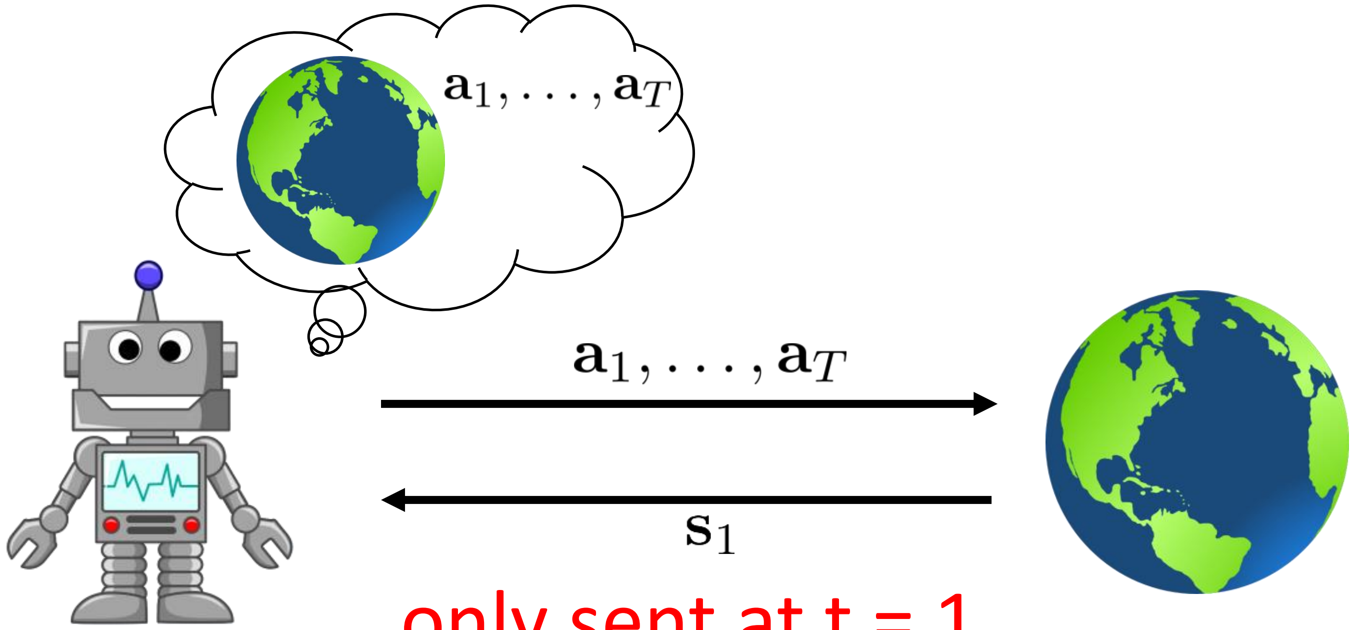
Aside: terminology

what is this “loop”?

closed-loop

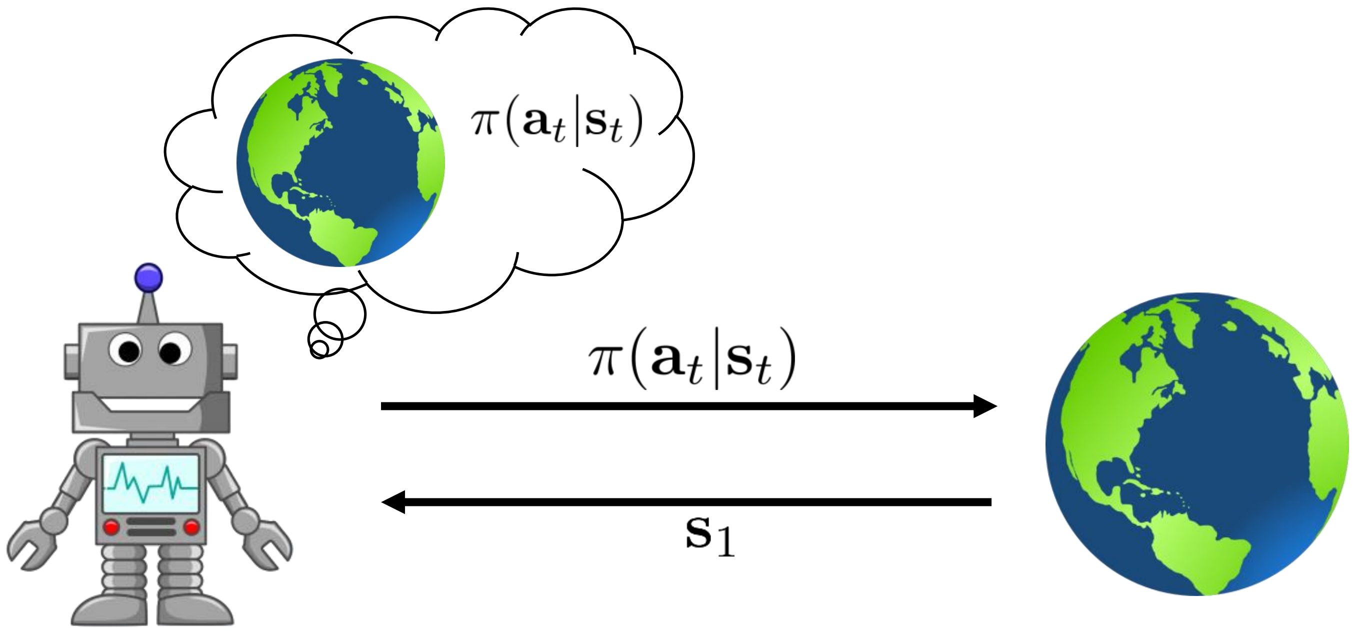


open-loop



only sent at t = 1,
then it’s one-way!

The stochastic **closed**-loop case



$$p(\mathbf{s}_1, \mathbf{a}_1, \ldots, \mathbf{s}_T, \mathbf{a}_T) = p(\mathbf{s}_1) \prod_{t=1}^{T} \pi(\mathbf{a}_t|\mathbf{s}_t) p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t)$$

$$\pi = \arg\max_{\pi} E_{\tau \sim p(\tau)} \left[\sum_t r(\mathbf{s}_t, \mathbf{a}_t) \right]$$

form of π?

neural net global

time-varying linear

$\mathbf{K}_t \mathbf{s}_t + \mathbf{k}_t$ local

(more on this later)

Stochastic optimization

abstract away optimal control/planning:

$$\mathbf{a}_1, \ldots, \mathbf{a}_T = \arg \max_{\mathbf{a}_1, \ldots, \mathbf{a}_T} \underbrace{J(\mathbf{a}_1, \ldots, \mathbf{a}_T)}_{\text{don't care what this is}}$$

$$\mathbf{A} = \arg\max_{\mathbf{A}} J(\mathbf{A})$$

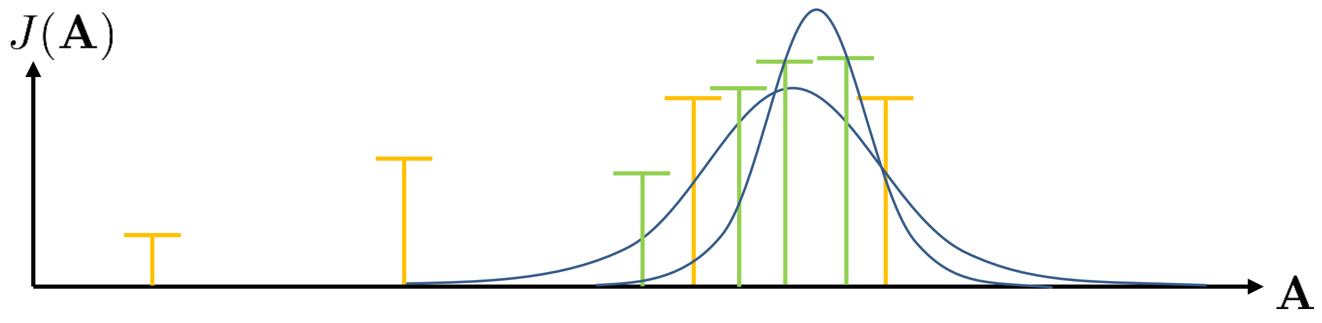
simplest method: guess & check "random shooting method"

1. pick $\mathbf{A}_1, \ldots, \mathbf{A}_N$ from some distribution (e.g., uniform)
2. choose $\mathbf{A}_i$ based on $\arg\max_i J(\mathbf{A}_i)$

Cross-entropy method (CEM)

1. pick $\mathbf{A}_1, \ldots, \mathbf{A}_N$ from some distribution (e.g., uniform)
2. choose $\mathbf{A}_i$ based on $\arg\max_i J(\mathbf{A}_i)$

can we do better?



typically use Gaussian distribution

cross-entropy method with continuous-valued inputs:

1. sample $\mathbf{A}_1, \ldots, \mathbf{A}_N$ from $p(\mathbf{A})$
2. evaluate $J(\mathbf{A}_1), \ldots, J(\mathbf{A}_N)$
3. pick the *elites* $\mathbf{A}_{i_1}, \ldots, \mathbf{A}_{i_M}$ with the highest value, where $M < N$
4. refit $p(\mathbf{A})$ to the elites $\mathbf{A}_{i_1}, \ldots, \mathbf{A}_{i_M}$

see also: CMA-ES (sort of like CEM with momentum)

What’s the upside?

1. Very fast if parallelized
2. Extremely simple

What’s the problem?

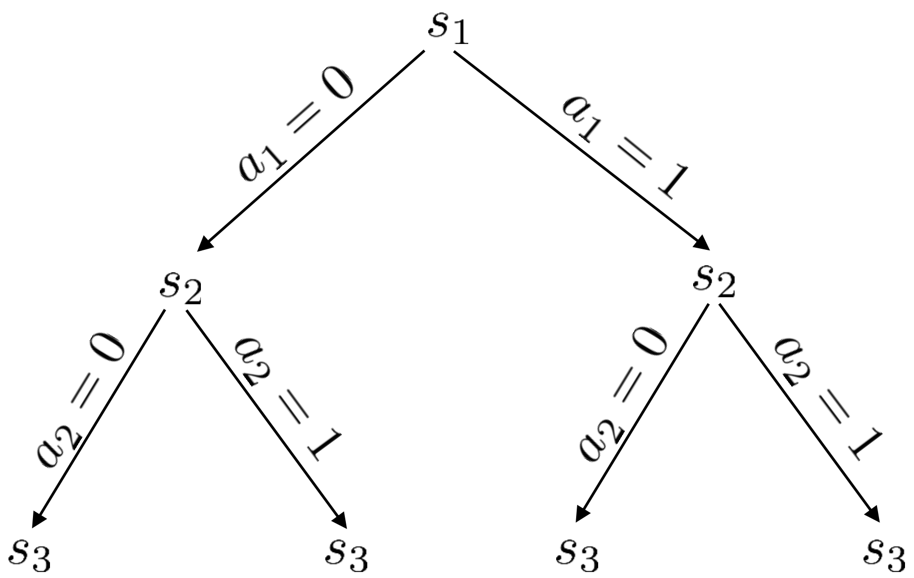
1. Very harsh dimensionality limit
2. Only open-loop planning

Discrete case: Monte Carlo tree search (MCTS)

s_t

a_t

discrete planning as a search problem

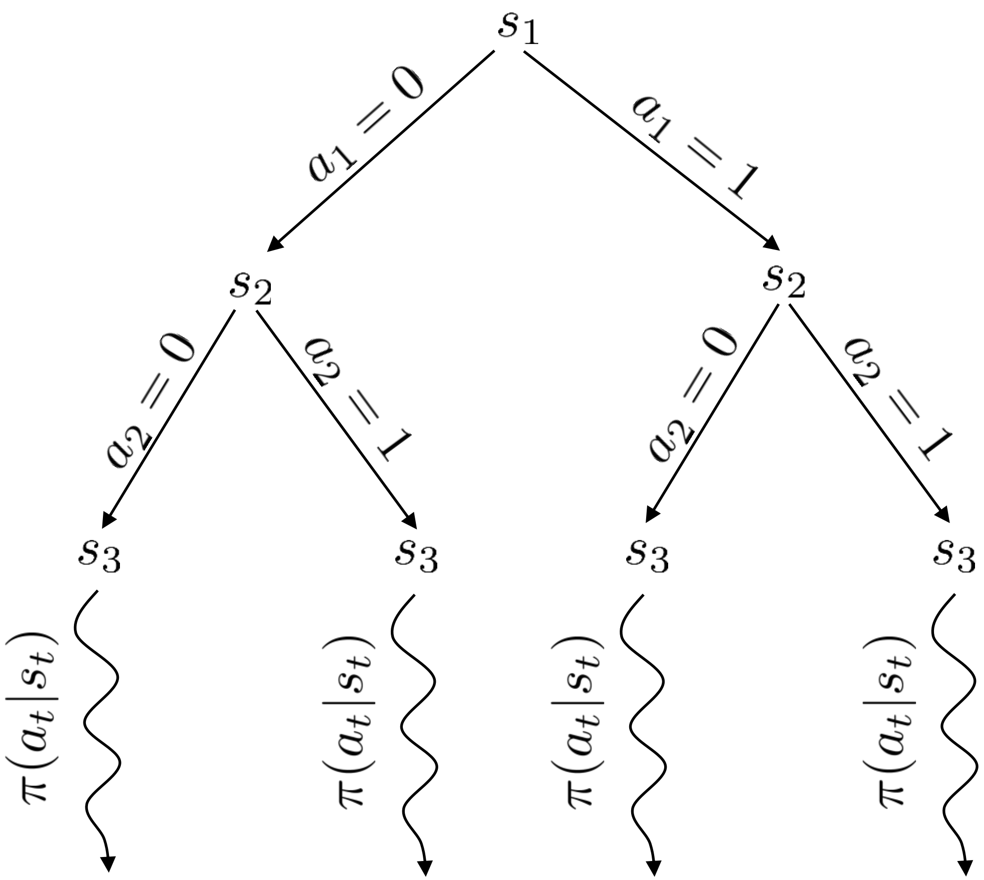


Discrete case: Monte Carlo tree search (MCTS)

how to approximate value without full tree?

s_t

a_t



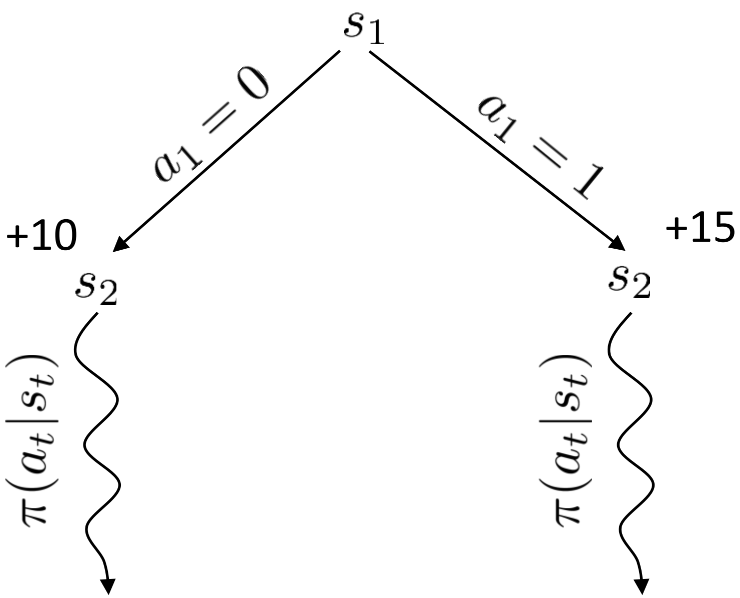
e.g., random policy

Discrete case: Monte Carlo tree search (MCTS)

can't search all paths – where to search first?

s_t

a_t



intuition: choose nodes with best reward, but also prefer rarely visited nodes

Discrete case: Monte Carlo tree search (MCTS)

generic MCTS sketch

1. find a leaf s_l using TreePolicy(s_1)
2. evaluate the leaf using DefaultPolicy(s_l)
3. update all values in tree between s_1 and s_l

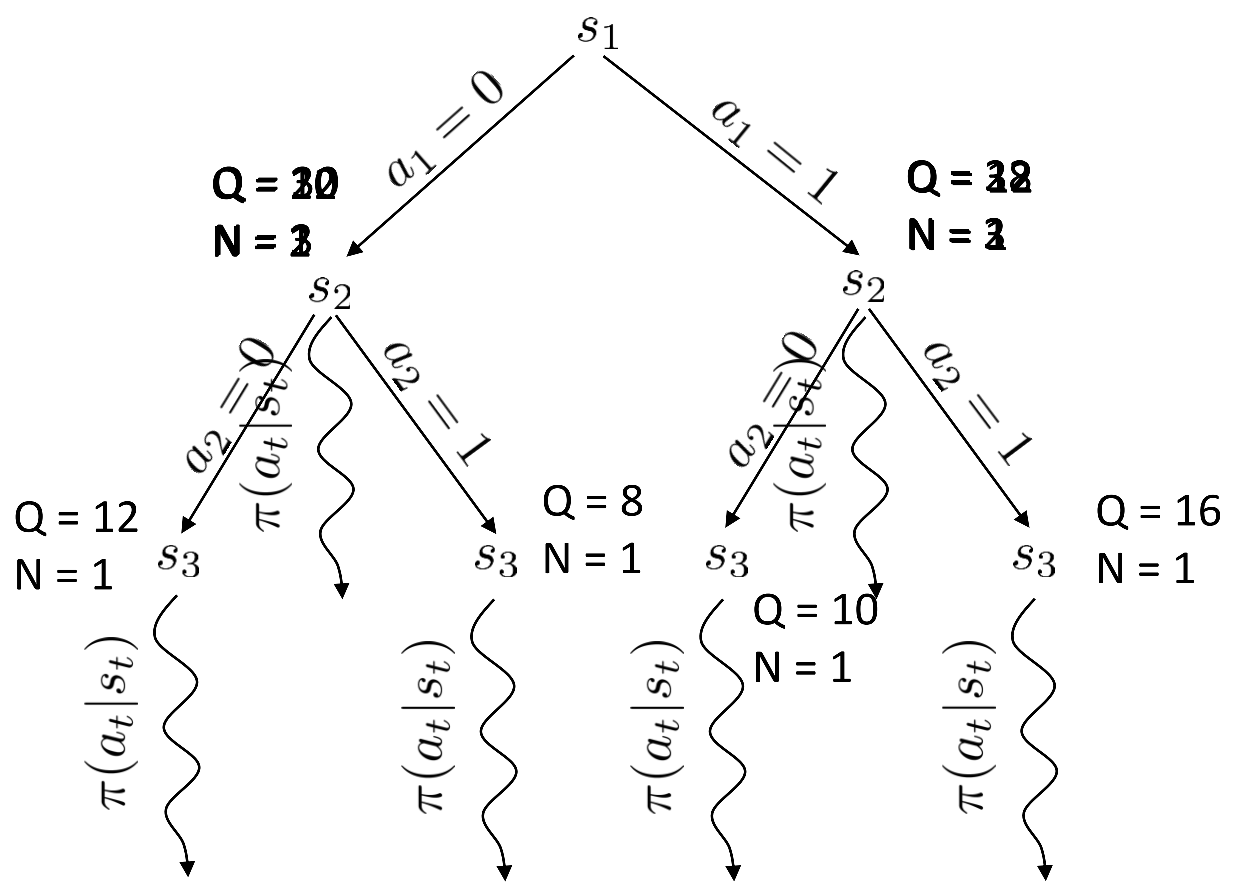
take best action from s_1

UCT TreePolicy(s_t)

if s_t not fully expanded, choose new a_t

else choose child with best Score(s_{t+1})

$$\text{Score}(s_t) = \frac{Q(s_t)}{N(s_t)} + 2C\sqrt{\frac{2\ln N(s_{t-1})}{N(s_t)}}$$



Additional reading

1. Browne, Powley, Whitehouse, Lucas, Cowling, Rohlfshagen, Tavener, Perez, Samothrakis, Colton. (2012). A Survey of Monte Carlo Tree Search Methods.
   - Survey of MCTS methods and basic summary.

Case study: imitation learning from MCTS

Deep Learning for Real-Time Atari Game Play Using Offline Monte-Carlo Tree Search Planning

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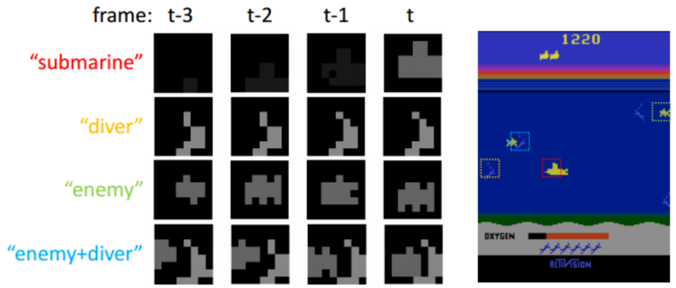


Case study: imitation learning from MCTS

DAgger

1. train $\pi_\theta(\mathbf{u}_t|\mathbf{o}_t)$ from human data $\mathcal{D} = \{\mathbf{o}_1, \mathbf{u}_1, \ldots, \mathbf{o}_N, \mathbf{u}_N\}$
2. run $\pi_\theta(\mathbf{u}_t|\mathbf{o}_t)$ to get dataset $\mathcal{D}_\pi = \{\mathbf{o}_1, \ldots, \mathbf{o}_M\}$
3. Ask human to label $\mathcal{D}_\pi$ with actions $\mathbf{u}_t$
3. Choose actions for states in $\mathcal{D}_\pi$ using MCTS
4. Aggregate: $\mathcal{D} \leftarrow \mathcal{D} \cup \mathcal{D}_\pi$

Why train a policy?

- In this case, MCTS is too slow for real-time play
- Other reasons – perception, generalization, etc.: more on this later

Break

Can we use derivatives?

$$\min_{\mathbf{u}_1,\ldots,\mathbf{u}_T} \sum_{t=1}^{T} c(\mathbf{x}_t, \mathbf{u}_t) \text{ s.t. } \mathbf{x}_t = f(\mathbf{x}_{t-1}, \mathbf{u}_{t-1})$$

$$\min_{\mathbf{u}_1,\ldots,\mathbf{u}_T} c(\mathbf{x}_1, \mathbf{u}_1) + c(f(\mathbf{x}_1, \mathbf{u}_1), \mathbf{u}_2) + \cdots + c(f(f(\ldots)\ldots), \mathbf{u}_T)$$

usual story: differentiate via backpropagation and optimize!

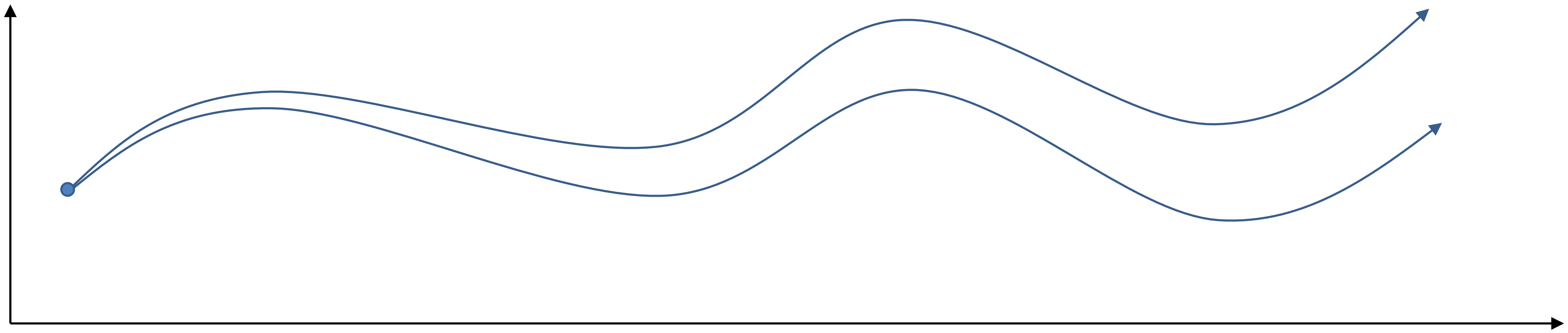
need $\frac{df}{d\mathbf{x}_t}, \frac{df}{d\mathbf{u}_t}, \frac{dc}{d\mathbf{x}_t}, \frac{dc}{d\mathbf{u}_t}$

in practice, it really helps to use a 2^{nd} order method!

Shooting methods vs collocation

shooting method: optimize over actions only

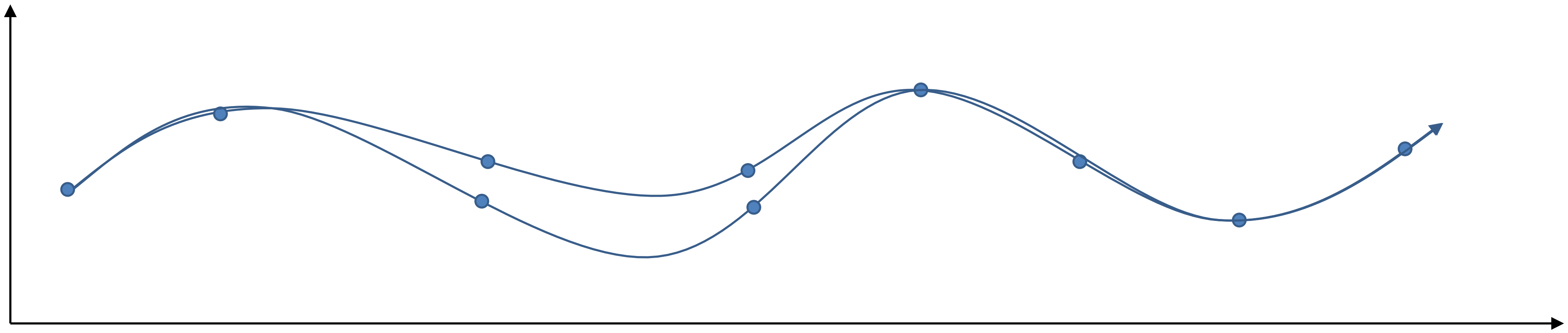
$$\min_{\mathbf{u}_1,\dots,\mathbf{u}_T} c(\mathbf{x}_1,\mathbf{u}_1) + c(f(\mathbf{x}_1,\mathbf{u}_1),\mathbf{u}_2) + \cdots + c(f(f(\dots)\dots),\mathbf{u}_T)$$



Shooting methods vs collocation

collocation method: optimize over actions and states, with constraints

$$\min_{\mathbf{u}_1,\ldots,\mathbf{u}_T,\mathbf{x}_1,\ldots,\mathbf{x}_T} \sum_{t=1}^{T} c(\mathbf{x}_t,\mathbf{u}_t) \text{ s.t. } \mathbf{x}_t = f(\mathbf{x}_{t-1},\mathbf{u}_{t-1})$$



Linear case: LQR

$$\min_{\mathbf{u}_1,\ldots,\mathbf{u}_T} c(\mathbf{x}_1,\mathbf{u}_1) + c(f(\mathbf{x}_1,\mathbf{u}_1),\mathbf{u}_2) + \cdots + c(f(f(\ldots)\ldots),\mathbf{u}_T)$$

$$f(\mathbf{x}_t,\mathbf{u}_t) = \mathbf{F}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \mathbf{f}_t$$

linear

$$c(\mathbf{x}_t,\mathbf{u}_t) = \frac{1}{2}\begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{C}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{c}_t$$

quadratic

Linear case: LQR

$$\min_{\mathbf{u}_1,\ldots,\mathbf{u}_T} c(\mathbf{x}_1,\mathbf{u}_1) + c(f(\mathbf{x}_1,\mathbf{u}_1),\mathbf{u}_2) + \cdots + \underbrace{c(\overbrace{f(f(\ldots)\ldots)}^{\mathbf{x}_T \text{ (unknown)}},\mathbf{u}_T)}_{\text{only term that depends on } \mathbf{u}_T}$$

$$c(\mathbf{x}_t,\mathbf{u}_t) = \frac{1}{2}\begin{bmatrix}\mathbf{x}_t\\ \mathbf{u}_t\end{bmatrix}^T \mathbf{C}_t \begin{bmatrix}\mathbf{x}_t\\ \mathbf{u}_t\end{bmatrix} + \begin{bmatrix}\mathbf{x}_t\\ \mathbf{u}_t\end{bmatrix}^T \mathbf{c}_t$$

$$f(\mathbf{x}_t,\mathbf{u}_t) = \mathbf{F}_t \begin{bmatrix}\mathbf{x}_t\\ \mathbf{u}_t\end{bmatrix} + \mathbf{f}_t$$

$$\mathbf{C}_T = \begin{bmatrix}\mathbf{C}_{\mathbf{x}_T,\mathbf{x}_T} & \mathbf{C}_{\mathbf{x}_T,\mathbf{u}_T}\\ \mathbf{C}_{\mathbf{u}_T,\mathbf{x}_T} & \mathbf{C}_{\mathbf{u}_T,\mathbf{u}_T}\end{bmatrix}$$

Base case: solve for $\mathbf{u}_T$ *only*

$$\mathbf{c}_T = \begin{bmatrix}\mathbf{c}_{\mathbf{x}_T}\\ \mathbf{c}_{\mathbf{u}_T}\end{bmatrix}$$

$$Q(\mathbf{x}_T,\mathbf{u}_T) = \text{const} + \frac{1}{2}\begin{bmatrix}\mathbf{x}_T\\ \mathbf{u}_T\end{bmatrix}^T \mathbf{C}_T \begin{bmatrix}\mathbf{x}_T\\ \mathbf{u}_T\end{bmatrix} + \begin{bmatrix}\mathbf{x}_T\\ \mathbf{u}_T\end{bmatrix}^T \mathbf{c}_T$$

$$\nabla_{\mathbf{u}_T} Q(\mathbf{x}_T,\mathbf{u}_T) = \mathbf{C}_{\mathbf{u}_T,\mathbf{x}_T}\mathbf{x}_T + \mathbf{C}_{\mathbf{u}_T,\mathbf{u}_T}\mathbf{u}_T + \mathbf{c}_{\mathbf{u}_T}^T = 0$$

$$\mathbf{K}_T = -\mathbf{C}_{\mathbf{u}_T,\mathbf{u}_T}^{-1}\mathbf{C}_{\mathbf{u}_T,\mathbf{x}_T}$$

$$\mathbf{u}_T = -\mathbf{C}_{\mathbf{u}_T,\mathbf{u}_T}^{-1}\left(\mathbf{C}_{\mathbf{u}_T,\mathbf{x}_T}\mathbf{x}_T + \mathbf{c}_{\mathbf{u}_T}\right) \qquad \mathbf{u}_T = \mathbf{K}_T\mathbf{x}_T + \mathbf{k}_T \qquad \mathbf{k}_T = -\mathbf{C}_{\mathbf{u}_T,\mathbf{u}_T}^{-1}\mathbf{c}_{\mathbf{u}_T}$$

Linear case: LQR

$$\mathbf{u}_T = \mathbf{K}_T \mathbf{x}_T + \mathbf{k}_T \qquad \mathbf{K}_T = -\mathbf{C}_{\mathbf{u}_T,\mathbf{u}_T}^{-1} \mathbf{C}_{\mathbf{u}_T,\mathbf{x}_T} \qquad \mathbf{k}_T = -\mathbf{C}_{\mathbf{u}_T,\mathbf{u}_T}^{-1} \mathbf{c}_{\mathbf{u}_T}$$

$$Q(\mathbf{x}_T, \mathbf{u}_T) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_T \\ \mathbf{u}_T \end{bmatrix}^T \mathbf{C}_T \begin{bmatrix} \mathbf{x}_T \\ \mathbf{u}_T \end{bmatrix} + \begin{bmatrix} \mathbf{x}_T \\ \mathbf{u}_T \end{bmatrix}^T \mathbf{c}_T$$

Since $\mathbf{u}_T$ is fully determined by $\mathbf{x}_T$, we can eliminate it via substitution!

$$V(\mathbf{x}_T) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_T \\ \mathbf{K}_T \mathbf{x}_T + \mathbf{k}_T \end{bmatrix}^T \mathbf{C}_T \begin{bmatrix} \mathbf{x}_T \\ \mathbf{K}_T \mathbf{x}_T + \mathbf{k}_T \end{bmatrix} + \begin{bmatrix} \mathbf{x}_T \\ \mathbf{K}_T \mathbf{x}_T + \mathbf{k}_T \end{bmatrix}^T \mathbf{c}_T$$

$$\begin{aligned} V(\mathbf{x}_T) = & \frac{1}{2} \mathbf{x}_T^T \mathbf{C}_{\mathbf{x}_T,\mathbf{x}_T} \mathbf{x}_T + \frac{1}{2} \mathbf{x}_T^T \mathbf{C}_{\mathbf{x}_T,\mathbf{u}_T} \mathbf{K}_T \mathbf{x}_T + \frac{1}{2} \mathbf{x}_T^T \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T,\mathbf{x}_T} \mathbf{x}_T + \frac{1}{2} \mathbf{x}_T^T \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T,\mathbf{u}_T} \mathbf{K}_T \mathbf{x}_T + \\ & \mathbf{x}_T^T \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T,\mathbf{u}_T} \mathbf{k}_T + \frac{1}{2} \mathbf{x}_T^T \mathbf{C}_{\mathbf{x}_T,\mathbf{u}_T} \mathbf{k}_T + \mathbf{x}_T^T \mathbf{c}_{\mathbf{x}_T} + \mathbf{x}_T^T \mathbf{K}_T^T \mathbf{c}_{\mathbf{u}_T} + \text{const} \end{aligned}$$

$$V(\mathbf{x}_T) = \text{const} + \frac{1}{2} \mathbf{x}_T^T \mathbf{V}_T \mathbf{x}_T + \mathbf{x}_T^T \mathbf{v}_T \qquad \mathbf{V}_T = \mathbf{C}_{\mathbf{x}_T,\mathbf{x}_T} + \mathbf{C}_{\mathbf{x}_T,\mathbf{u}_T} \mathbf{K}_T + \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T,\mathbf{x}_T} + \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T,\mathbf{u}_T} \mathbf{K}_T$$

$$\mathbf{v}_T = \mathbf{c}_{\mathbf{x}_T} + \mathbf{C}_{\mathbf{x}_T,\mathbf{u}_T} \mathbf{k}_T + \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T} + \mathbf{K}_T^T \mathbf{C}_{\mathbf{u}_T,\mathbf{u}_T} \mathbf{k}_T$$

Linear case: LQR

Solve for $\mathbf{u}_{T-1}$ in terms of $\mathbf{x}_{T-1}$

$\mathbf{u}_{T-1}$ affects $\mathbf{x}_T$!

$$f(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}) = \mathbf{x}_T = \mathbf{F}_{T-1} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix} + \mathbf{f}_{T-1}$$

$$Q(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \mathbf{C}_{T-1} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix} + \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \mathbf{c}_{T-1} + V(f(\mathbf{x}_{T-1}, \mathbf{u}_{T-1}))$$

$$V(\mathbf{x}_T) = \text{const} + \frac{1}{2}\mathbf{x}_T^T \mathbf{V}_T \mathbf{x}_T + \mathbf{x}_T^T \mathbf{v}_T$$

$$V(\mathbf{x}_T) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \underbrace{\mathbf{F}_{T-1}^T \mathbf{V}_T \mathbf{F}_{T-1}}_{\text{quadratic}} \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix} + \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \underbrace{\mathbf{F}_{T-1}^T \mathbf{V}_T \mathbf{f}_{T-1}}_{\text{linear}} + \begin{bmatrix} \mathbf{x}_{T-1} \\ \mathbf{u}_{T-1} \end{bmatrix}^T \underbrace{\mathbf{F}_{T-1}^T \mathbf{v}_T}_{\text{linear}}$$

Linear case: LQR

$$Q(\mathbf{x}_{T-1},\mathbf{u}_{T-1}) = \text{const} + \frac{1}{2}\begin{bmatrix}\mathbf{x}_{T-1}\\ \mathbf{u}_{T-1}\end{bmatrix}^T \mathbf{C}_{T-1}\begin{bmatrix}\mathbf{x}_{T-1}\\ \mathbf{u}_{T-1}\end{bmatrix} + \begin{bmatrix}\mathbf{x}_{T-1}\\ \mathbf{u}_{T-1}\end{bmatrix}^T \mathbf{c}_{T-1} + V(f(\mathbf{x}_{T-1},\mathbf{u}_{T-1}))$$

$$V(\mathbf{x}_T) = \text{const} + \frac{1}{2}\begin{bmatrix}\mathbf{x}_{T-1}\\ \mathbf{u}_{T-1}\end{bmatrix}^T \underbrace{\mathbf{F}_{T-1}^T\mathbf{V}_T\mathbf{F}_{T-1}}\begin{bmatrix}\mathbf{x}_{T-1}\\ \mathbf{u}_{T-1}\end{bmatrix} + \begin{bmatrix}\mathbf{x}_{T-1}\\ \mathbf{u}_{T-1}\end{bmatrix}^T \underbrace{\mathbf{F}_{T-1}^T\mathbf{V}_T\mathbf{f}_{T-1}} + \begin{bmatrix}\mathbf{x}_{T-1}\\ \mathbf{u}_{T-1}\end{bmatrix}^T \underbrace{\mathbf{F}_{T-1}^T\mathbf{v}_T}$$

quadratic linear linear

$$Q(\mathbf{x}_{T-1},\mathbf{u}_{T-1}) = \text{const} + \frac{1}{2}\begin{bmatrix}\mathbf{x}_{T-1}\\ \mathbf{u}_{T-1}\end{bmatrix}^T \mathbf{Q}_{T-1}\begin{bmatrix}\mathbf{x}_{T-1}\\ \mathbf{u}_{T-1}\end{bmatrix} + \begin{bmatrix}\mathbf{x}_{T-1}\\ \mathbf{u}_{T-1}\end{bmatrix}^T \mathbf{q}_{T-1}$$

$$\mathbf{Q}_{T-1} = \mathbf{C}_{T-1} + \mathbf{F}_{T-1}^T\mathbf{V}_T\mathbf{F}_{T-1}$$

$$\mathbf{q}_{T-1} = \mathbf{c}_{T-1} + \mathbf{F}_{T-1}^T\mathbf{V}_T\mathbf{f}_{T-1} + \mathbf{F}_{T-1}^T\mathbf{v}_T$$

$$\nabla_{\mathbf{u}_{T-1}} Q(\mathbf{x}_{T-1},\mathbf{u}_{T-1}) = \mathbf{Q}_{\mathbf{u}_{T-1},\mathbf{x}_{T-1}}\mathbf{x}_{T-1} + \mathbf{Q}_{\mathbf{u}_{T-1},\mathbf{u}_{T-1}}\mathbf{u}_{T-1} + \mathbf{q}_{\mathbf{u}_{T-1}}^T = 0$$

$$\mathbf{u}_{T-1} = \mathbf{K}_{T-1}\mathbf{x}_{T-1} + \mathbf{k}_{T-1} \qquad \mathbf{K}_{T-1} = -\mathbf{Q}_{\mathbf{u}_{T-1},\mathbf{u}_{T-1}}^{-1}\mathbf{Q}_{\mathbf{u}_{T-1},\mathbf{x}_{T-1}} \qquad \mathbf{k}_{T-1} = -\mathbf{Q}_{\mathbf{u}_{T-1},\mathbf{u}_{T-1}}^{-1}\mathbf{q}_{\mathbf{u}_{T-1}}$$

Linear case: LQR

Backward recursion

for $t = T$ to 1:

$$\mathbf{Q}_t = \mathbf{C}_t + \mathbf{F}_t^T \mathbf{V}_{t+1} \mathbf{F}_t$$

$$\mathbf{q}_t = \mathbf{c}_t + \mathbf{F}_t^T \mathbf{V}_{t+1} \mathbf{f}_t + \mathbf{F}_t^T \mathbf{v}_{t+1}$$

$$Q(\mathbf{x}_t, \mathbf{u}_t) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{Q}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{q}_t$$

$$\mathbf{u}_t \leftarrow \arg\min_{\mathbf{u}_t} Q(\mathbf{x}_t, \mathbf{u}_t) = \mathbf{K}_t \mathbf{x}_t + \mathbf{k}_t$$

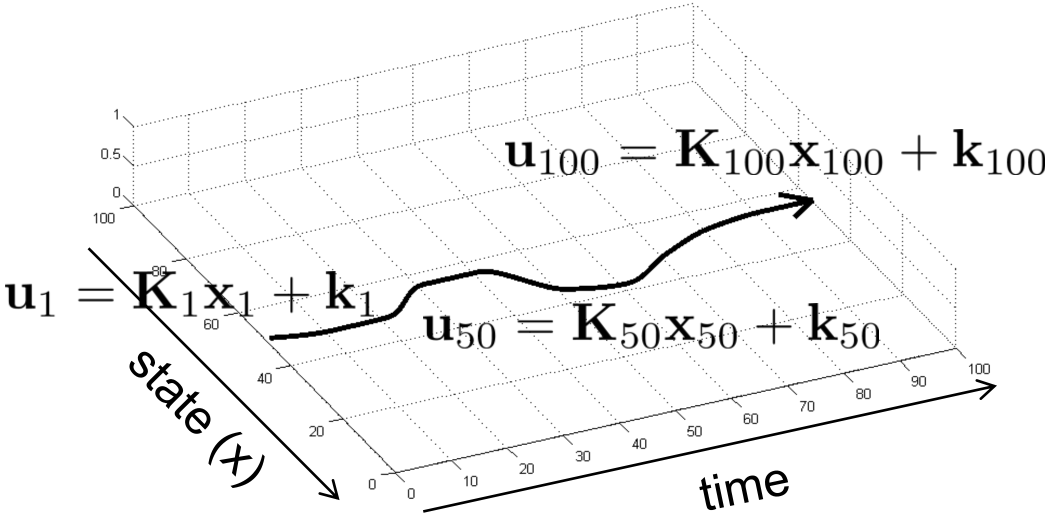
$$\mathbf{K}_t = -\mathbf{Q}_{\mathbf{u}_t,\mathbf{u}_t}^{-1} \mathbf{Q}_{\mathbf{u}_t,\mathbf{x}_t}$$

$$\mathbf{k}_t = -\mathbf{Q}_{\mathbf{u}_t,\mathbf{u}_t}^{-1} \mathbf{q}_{\mathbf{u}_t}$$

$$\mathbf{V}_t = \mathbf{Q}_{\mathbf{x}_t,\mathbf{x}_t} + \mathbf{Q}_{\mathbf{x}_t,\mathbf{u}_t} \mathbf{K}_t + \mathbf{K}_t^T \mathbf{Q}_{\mathbf{u}_t,\mathbf{x}_t} + \mathbf{K}_t^T \mathbf{Q}_{\mathbf{u}_t,\mathbf{u}_t} \mathbf{K}_t$$

$$\mathbf{v}_t = \mathbf{q}_{\mathbf{x}_t} + \mathbf{Q}_{\mathbf{x}_t,\mathbf{u}_t} \mathbf{k}_t + \mathbf{K}_t^T \mathbf{Q}_{\mathbf{u}_t} + \mathbf{K}_t^T \mathbf{Q}_{\mathbf{u}_t,\mathbf{u}_t} \mathbf{k}_t$$

$$V(\mathbf{x}_t) = \text{const} + \frac{1}{2} \mathbf{x}_t^T \mathbf{V}_t \mathbf{x}_t + \mathbf{x}_t^T \mathbf{v}_t$$



we know $\mathbf{x}_1$!

Forward recursion

for $t = 1$ to T:

$$\mathbf{u}_t = \mathbf{K}_t \mathbf{x}_t + \mathbf{k}_t$$

$$\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t)$$

Some useful definitions

Backward recursion

for $t = T$ to 1:

$$\mathbf{Q}_t = \mathbf{C}_t + \mathbf{F}_t^T \mathbf{V}_{t+1} \mathbf{F}_t$$

$$\mathbf{q}_t = \mathbf{c}_t + \mathbf{F}_t^T \mathbf{V}_{t+1} \mathbf{f}_t + \mathbf{F}_t^T \mathbf{v}_{t+1}$$

$$Q(\mathbf{x}_t, \mathbf{u}_t) = \text{const} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{Q}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{q}_t$$

total cost from now until end if we take $\mathbf{u}_t$ from state $\mathbf{x}_t$

$$\mathbf{u}_t \leftarrow \arg\min_{\mathbf{u}_t} Q(\mathbf{x}_t, \mathbf{u}_t) = \mathbf{K}_t \mathbf{x}_t + \mathbf{k}_t$$

$$\mathbf{K}_t = -\mathbf{Q}_{\mathbf{u}_t,\mathbf{u}_t}^{-1} \mathbf{Q}_{\mathbf{u}_t,\mathbf{x}_t}$$

$$\mathbf{k}_t = -\mathbf{Q}_{\mathbf{u}_t,\mathbf{u}_t}^{-1} \mathbf{q}_{\mathbf{u}_t}$$

$$\mathbf{V}_t = \mathbf{Q}_{\mathbf{x}_t,\mathbf{x}_t} + \mathbf{Q}_{\mathbf{x}_t,\mathbf{u}_t} \mathbf{K}_t + \mathbf{K}_t^T \mathbf{Q}_{\mathbf{u}_t,\mathbf{x}_t} + \mathbf{K}_t^T \mathbf{Q}_{\mathbf{u}_t,\mathbf{u}_t} \mathbf{K}_t$$

$$\mathbf{v}_t = \mathbf{q}_{\mathbf{x}_t} + \mathbf{Q}_{\mathbf{x}_t,\mathbf{u}_t} \mathbf{k}_t + \mathbf{K}_t^T \mathbf{Q}_{\mathbf{u}_t} + \mathbf{K}_t^T \mathbf{Q}_{\mathbf{u}_t,\mathbf{u}_t} \mathbf{k}_t$$

$$V(\mathbf{x}_t) = \text{const} + \frac{1}{2} \mathbf{x}_t^T \mathbf{V}_t \mathbf{x}_t + \mathbf{x}_t^T \mathbf{v}_t$$

total cost from now until end from state $\mathbf{x}_t$

$$V(\mathbf{x}_t) = \min_{\mathbf{u}_t} Q(\mathbf{x}_t, \mathbf{u}_t)$$

Stochastic dynamics

$$f(\mathbf{x}_t, \mathbf{u}_t) = \mathbf{F}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \mathbf{f}_t$$

$$\mathbf{x}_{t+1} \sim p(\mathbf{x}_{t+1} | \mathbf{x}_t, \mathbf{u}_t)$$

$$p(\mathbf{x}_{t+1} | \mathbf{x}_t, \mathbf{u}_t) = \mathcal{N}\left(\mathbf{F}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \mathbf{f}_t, \Sigma_t\right)$$

Solution: choose actions according to $\mathbf{u}_t = \mathbf{K}_t \mathbf{x}_t + \mathbf{k}_t$

$\mathbf{x}_t \sim p(\mathbf{x}_t)$, no longer deterministic, but $p(\mathbf{x}_t)$ is Gaussian

no change to algorithm! can ignore Σ_t due to symmetry of Gaussians (checking this is left as an exercise; hint: the expectation of a quadratic under a Gaussian has an analytic solution)

Nonlinear case: DDP/iterative LQR

Linear-quadratic assumptions:

$$f(\mathbf{x}_t, \mathbf{u}_t) = \mathbf{F}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \mathbf{f}_t$$

$$c(\mathbf{x}_t, \mathbf{u}_t) = \frac{1}{2} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{C}_t \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^T \mathbf{c}_t$$

Can we *approximate* a nonlinear system as a linear-quadratic system?

$$f(\mathbf{x}_t, \mathbf{u}_t) \approx f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) + \nabla_{\mathbf{x}_t, \mathbf{u}_t} f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix}$$

$$c(\mathbf{x}_t, \mathbf{u}_t) \approx c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) + \nabla_{\mathbf{x}_t, \mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix}^T \nabla^2_{\mathbf{x}_t, \mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix}$$

Nonlinear case: DDP/iterative LQR

$$f(\mathbf{x}_t, \mathbf{u}_t) \approx f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) + \nabla_{\mathbf{x}_t, \mathbf{u}_t} f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix}$$

$$c(\mathbf{x}_t, \mathbf{u}_t) \approx c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) + \nabla_{\mathbf{x}_t, \mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix}^T \nabla^2_{\mathbf{x}_t, \mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t) \begin{bmatrix} \mathbf{x}_t - \hat{\mathbf{x}}_t \\ \mathbf{u}_t - \hat{\mathbf{u}}_t \end{bmatrix}$$

$$\bar{f}(\delta\mathbf{x}_t, \delta\mathbf{u}_t) = \underbrace{\mathbf{F}_t}_{\nabla_{\mathbf{x}_t, \mathbf{u}_t} f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)} \begin{bmatrix} \delta\mathbf{x}_t \\ \delta\mathbf{u}_t \end{bmatrix}$$

$$\bar{c}(\delta\mathbf{x}_t, \delta\mathbf{u}_t) = \frac{1}{2} \begin{bmatrix} \delta\mathbf{x}_t \\ \delta\mathbf{u}_t \end{bmatrix}^T \underbrace{\mathbf{C}_t}_{\nabla^2_{\mathbf{x}_t, \mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)} \begin{bmatrix} \delta\mathbf{x}_t \\ \delta\mathbf{u}_t \end{bmatrix} + \begin{bmatrix} \delta\mathbf{x}_t \\ \delta\mathbf{u}_t \end{bmatrix}^T \underbrace{\mathbf{c}_t}_{\nabla_{\mathbf{x}_t, \mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)}$$

$$\delta\mathbf{x}_t = \mathbf{x}_t - \hat{\mathbf{x}}_t$$

$$\delta\mathbf{u}_t = \mathbf{u}_t - \hat{\mathbf{u}}_t$$

Now we can run LQR with dynamics $\bar{f}$, cost $\bar{c}$, state $\delta\mathbf{x}_t$, and action $\delta\mathbf{u}_t$

Nonlinear case: DDP/iterative LQR

Iterative LQR (simplified pseudocode)

until convergence:

$\mathbf{F}_t = \nabla_{\mathbf{x}_t,\mathbf{u}_t} f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)$

$\mathbf{c}_t = \nabla_{\mathbf{x}_t,\mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)$

$\mathbf{C}_t = \nabla^2_{\mathbf{x}_t,\mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)$

Run LQR backward pass on state $\delta\mathbf{x}_t = \mathbf{x}_t - \hat{\mathbf{x}}_t$ and action $\delta\mathbf{u}_t = \mathbf{u}_t - \hat{\mathbf{u}}_t$

Run forward pass with real nonlinear dynamics and $\mathbf{u}_t = \mathbf{K}_t(\mathbf{x}_t - \hat{\mathbf{x}}_t) + \mathbf{k}_t + \hat{\mathbf{u}}_t$

Update $\hat{\mathbf{x}}_t$ and $\hat{\mathbf{u}}_t$ based on states and actions in forward pass

Nonlinear case: DDP/iterative LQR

Why does this work?

Compare to Newton's method for computing $\min_{\mathbf{x}} g(\mathbf{x})$:

until convergence:

$$\mathbf{g} = \nabla_{\mathbf{x}} g(\hat{\mathbf{x}})$$

$$\mathbf{H} = \nabla^2_{\mathbf{x}} g(\hat{\mathbf{x}})$$

$$\hat{\mathbf{x}} \leftarrow \arg\min_{\mathbf{x}} \frac{1}{2}(\mathbf{x} - \hat{\mathbf{x}})^T \mathbf{H}(\mathbf{x} - \hat{\mathbf{x}}) + \mathbf{g}^T(\mathbf{x} - \hat{\mathbf{x}})$$

Iterative LQR (iLQR) is the same idea: locally approximate a complex nonlinear function via Taylor expansion

In fact, iLQR is an approximation of Newton's method for solving

$$\min_{\mathbf{u}_1,\dots,\mathbf{u}_T} c(\mathbf{x}_1, \mathbf{u}_1) + c(f(\mathbf{x}_1, \mathbf{u}_1), \mathbf{u}_2) + \cdots + c(f(f(\dots)\dots), \mathbf{u}_T)$$

Nonlinear case: DDP/iterative LQR

In fact, iLQR is an approximation of Newton's method for solving

$$\min_{\mathbf{u}_1,\dots,\mathbf{u}_T} c(\mathbf{x}_1,\mathbf{u}_1) + c(f(\mathbf{x}_1,\mathbf{u}_1),\mathbf{u}_2) + \cdots + c(f(f(\dots)\dots),\mathbf{u}_T)$$

To get Newton's method, need to use *second order* dynamics approximation:

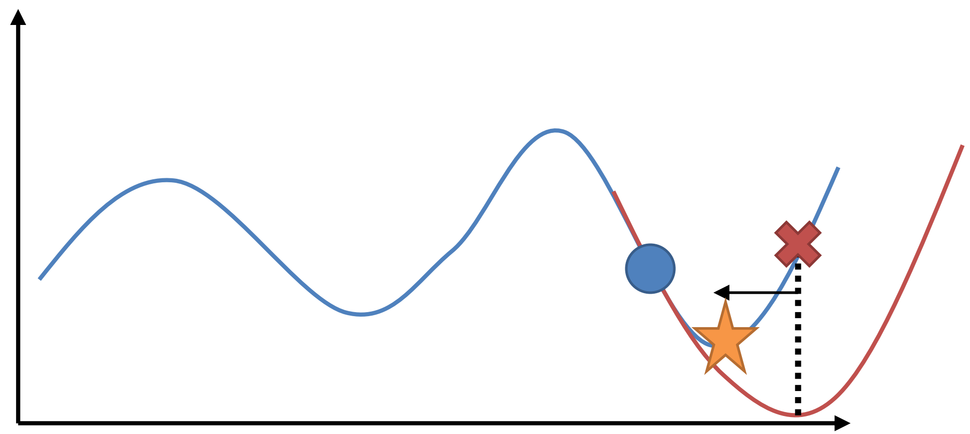
$$f(\mathbf{x}_t,\mathbf{u}_t) \approx f(\hat{\mathbf{x}}_t,\hat{\mathbf{u}}_t) + \nabla_{\mathbf{x}_t,\mathbf{u}_t} f(\hat{\mathbf{x}}_t,\hat{\mathbf{u}}_t)\begin{bmatrix}\delta\mathbf{x}_t\\ \delta\mathbf{u}_t\end{bmatrix} + \frac{1}{2}\left(\nabla^2_{\mathbf{x}_t,\mathbf{u}_t} f(\hat{\mathbf{x}}_t,\hat{\mathbf{u}}_t)\cdot\begin{bmatrix}\delta\mathbf{x}_t\\ \delta\mathbf{u}_t\end{bmatrix}\right)\begin{bmatrix}\delta\mathbf{x}_t\\ \delta\mathbf{u}_t\end{bmatrix}$$

differential dynamic programming (DDP)

Nonlinear case: DDP/iterative LQR

$$\hat{\mathbf{x}} \leftarrow \arg\min_{\mathbf{x}} \frac{1}{2}(\mathbf{x} - \hat{\mathbf{x}})^T \mathbf{H}(\mathbf{x} - \hat{\mathbf{x}}) + \mathbf{g}^T(\mathbf{x} - \hat{\mathbf{x}})$$

why is this a bad idea?



until convergence:

$$\mathbf{F}_t = \nabla_{\mathbf{x}_t,\mathbf{u}_t} f(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)$$

$$\mathbf{c}_t = \nabla_{\mathbf{x}_t,\mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)$$

$$\mathbf{C}_t = \nabla^2_{\mathbf{x}_t,\mathbf{u}_t} c(\hat{\mathbf{x}}_t, \hat{\mathbf{u}}_t)$$

Run LQR backward pass on state $\delta\mathbf{x}_t = \mathbf{x}_t - \hat{\mathbf{x}}_t$ and action $\delta\mathbf{u}_t = \mathbf{u}_t - \hat{\mathbf{u}}_t$

Run forward pass with $\mathbf{u}_t = \mathbf{K}_t(\mathbf{x}_t - \hat{\mathbf{x}}_t) + \alpha\mathbf{k}_t + \hat{\mathbf{u}}_t$

Update $\hat{\mathbf{x}}_t$ and $\hat{\mathbf{u}}_t$ based on states and actions in forward pass

search over α
until improvement achieved

Additional reading

1. Mayne, Jacobson. (1970). Differential dynamic programming.
   - Original differential dynamic programming algorithm.
2. Tassa, Erez, Todorov. (2012). Synthesis and Stabilization of Complex Behaviors through Online Trajectory Optimization.
   - Practical guide for implementing non-linear iterative LQR.
3. Levine, Abbeel. (2014). Learning Neural Network Policies with Guided Policy Search under Unknown Dynamics.
   - Probabilistic formulation and trust region alternative to deterministic line search.

Case study: nonlinear model-predictive control

Synthesis and Stabilization of Complex Behaviors through Online Trajectory Optimization

Yuval Tassa, Tom Erez and Emanuel Todorov
University of Washington

every time step:

observe the state $\mathbf{x}_t$

use iLQR to plan $\mathbf{u}_t, \ldots, \mathbf{u}_T$ to minimize $\sum_{t'=t}^{t+T} c(\mathbf{x}_{t'}, \mathbf{u}_{t'})$

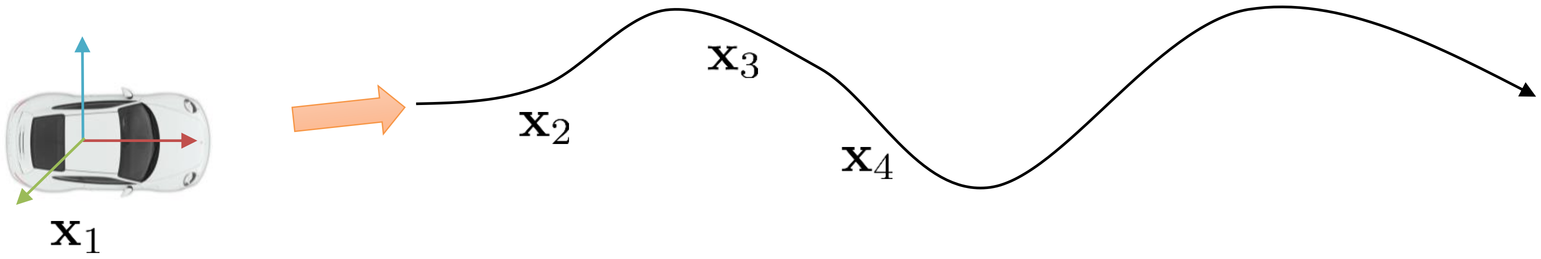
execute action $\mathbf{u}_t$, discard $\mathbf{u}_{t+1}, \ldots, \mathbf{u}_{t+T}$

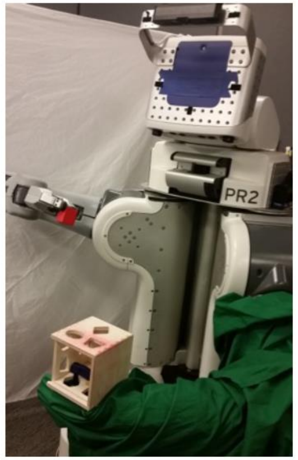
Synthesis of Complex Behaviors with Online Trajectory Optimization

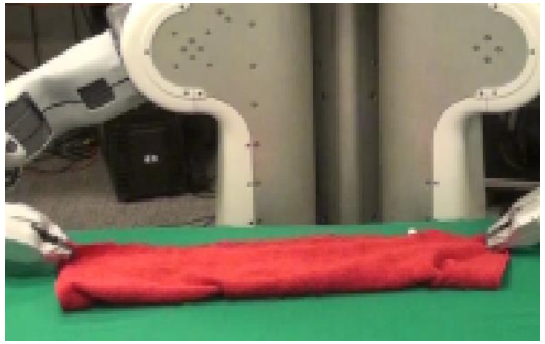
Yuval Tassa, Tom Erez & Emo Todorov

IEEE International Conference
on Intelligent Robots and Systems
2012

What's wrong with known dynamics?







Next time: learning the dynamics model