Markov Decision Processes and Solving Finite Problems

February 8, 2017

Overview of Upcoming Lectures

Feb 8:	Markov decision processes, value iteration, policy iteration
Feb 13:	Policy gradients
Feb 15:	Learning Q-functions: Q-learning, SARSA, and others
Feb 22:	Advanced Q-functions: replay buffers, target networks, double Q-learning
next...	Advanced model learning and imitation learning
next...	Advanced policy gradient methods, and the exploration problem

Overview for This Lecture

- This lecture assumes you have a known system with a finite number of states and actions.
- How to exactly solve for optimal policy
  - Value iteration
  - Policy iteration
  - Modified policy iteration

How Does This Lecture Fit In?

- Value Iteration $\xrightarrow{\text{small updates + neural nets}}$ deep Q-network methods
- Policy Iteration $\xrightarrow{\text{small updates + neural nets}}$ deep policy gradient methods

Markov Decision Process

Defined by the following components:

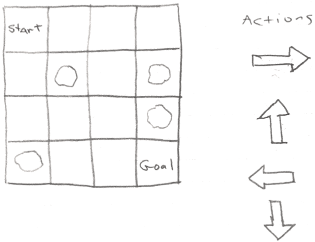
- $\mathcal{S}$: **state space**, a set of states of the environment.
- $\mathcal{A}$: **action space**, a set of actions, which the agent selects from at each timestep.
- $P(r, s' \mid s, a)$: a transition probability distribution.
  - Alternatively, $P(s' \mid s, a)$ and one of $R(s)$, $R(s, a)$ or $R(s, a, s')$

Partially Observed MDPs

- Instead of observing full state s, agent observes y, with $y \sim P(y \mid s)$.
- A MDP can be trivially mapped onto a POMDP
- A POMDP can be mapped onto an MDP:

$$\tilde{s}_0 = \{y_0\}, \quad \tilde{s}_1 = \{y_0, y_1\}, \quad \tilde{s}_2 = \{y_0, y_1, y_2\}, \ldots$$

Simple MDP: Frozen Lake



- Gym: FrozenLake-v0
- START state, GOAL state, other locations are FROZEN (safe) or HOLE (unsafe).
- Episode terminates when GOAL or HOLE state is reached
- Receive reward=1 when entering GOAL, 0 otherwise
- 4 directions are actions, but you move in wrong direction with probability 0.5.

Policies

- Deterministic policies $a = \pi(s)$
- Stochastic policies $a \sim \pi(a \mid s)$

Problems Involving MDPs

- **Policy optimization**: maximize expected reward with respect to policy π

$$\underset{\pi}{\text{maximize}}\, \mathbb{E}\left[\sum_{t=0}^{\infty} r_t\right]$$

- **Policy evaluation**: compute expected **return** for fixed policy π
  - return := sum of future rewards in an episode (i.e., a trajectory)
    - Discounted return: $r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \dots$
    - Undiscounted return: $r_t + r_{t+1} + \cdots + r_{T-1} + V(s_T)$
  - Performance of policy:
    $\eta(\pi) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t\right]$
  - State value function:
    $V^\pi(s) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t \mid s_0 = s\right]$
  - State-action value function:
    $Q^\pi(s, a) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t \mid s_0 = s, a_0 = a\right]$

Value Iteration: Finite Horizon Case

- Problem:

$$\max_{\pi_0} \max_{\pi_1} \dots \max_{\pi_{T-1}} \mathbb{E}\left[r_0 + r_1 + \cdots + r_{T-1} + V_T(s_T)\right]$$

- Swap maxes and expectations:

$$\max_{\pi_0} \mathbb{E}\left[r_0 + \max_{\pi_1} \mathbb{E}\left[r_1 + \cdots + \max_{\pi_{T-1}} \mathbb{E}\left[r_{T-1} + V_T(s_T)\right]\right]\right]$$

- Solve innermost problem: for each $s \in \mathcal{S}$

$$\pi_{T-1}(s), V_{T-1}(s) = \underset{a}{\text{maximize}}\, \mathbb{E}_{s_T}\left[r_{T-1} + V_T(s_T)\right]$$

- Original problem becomes

$$\max_{\pi_0} \mathbb{E}\left[r_0 + \max_{\pi_1} \mathbb{E}\left[r_1 + \cdots + \underbrace{\max_{\pi_{T-1}} \mathbb{E}\left[r_{T-1} + V_T(s_T)\right]}_{V_{T-1}(s)}\right]\right]$$

$$\max_{\pi_0} \mathbb{E}\left[r_0 + \max_{\pi_1} \mathbb{E}\left[r_1 + \cdots + \max_{\pi_{T-2}} \mathbb{E}\left[r_{T-2} + V_{T-1}(s_{T-1})\right]\right]\right]$$

Value Iteration: Finite Horizon Case

Algorithm 1 Finite Horizon Value Iteration

for $t = T-1, T-2, \ldots, 0$ **do**
    for $s \in \mathcal{S}$ **do**
        $\pi_t(s), V_t(s) = \text{maximize}_a \, \mathbb{E}\left[r_t + V_{t+1}(s_{t+1})\right]$
    end for
end for

Discounted Setting

- Discount factor $\gamma \in [0, 1)$, downweights future rewards
- Discounted return: $r_0 + \gamma r_1 + \gamma^2 r_2 + \dots$
- Coefficients $(1, \gamma, \gamma^2, \dots) \Rightarrow$ informally, we're adding up $1 + \gamma + \gamma^2 + \dots = 1/(1-\gamma)$ timesteps. *Effective time horizon* $1/(1-\gamma)$.
- Want to solve for policy that'll optimize discounted sum of rewards from each state.
- Discounted problem can be obtained by adding transitions to "sink state" (where agent gets stuck and receives zero reward)

$$\tilde{P}(s' \mid s, a) = \begin{cases} P(s' \mid s, a) \text{ with probability } \gamma \\ \text{sink state with probability } 1 - \gamma \end{cases}$$

Infinite-Horizon VI Via Finite-Horizon VI

- $\underset{\pi_0\pi_1\pi_2\ldots}{\text{maximize}}\, \mathbb{E}\left[r_0 + \gamma r_1 + \gamma^2 r_2 + \ldots\right]$
- Can rewrite with nested sum

$$\max_{\pi_0} \mathbb{E}\left[r_0 + \gamma \max_{\pi_1} \mathbb{E}\left[r_1 + \gamma \max_{\pi_2} \mathbb{E}\left[r_2 + \ldots\right]\right]\right]$$

- Pretend there's finite horizon T, ignore $r_T, r_{T+1}, \ldots$
  - error $\epsilon \leq r_{\max}\gamma^T/(1-\gamma)$
  - resulting nonstationary policy only suboptimal by ϵ
  - π_0, V_0 converges to optimal policy as $T \to \infty$.

Infinite-Horizon VI

Algorithm 2 Infinite-Horizon Value Iteration

Initialize $V^{(0)}$ arbitrarily.
for $n = 0, 1, 2, \ldots$ until termination condition **do**
  for $s \in \mathcal{S}$ **do**

$$\pi^{(n+1)}(s), V^{(n+1)}(s) = \text{maximize}_a\, E_{s_T}[r_{T-1} + \gamma V^{(n)}(s_T)]$$

  end for
end for

Note that $V^{(n)}$ is exactly V_0 in a finite-horizon problem with n timesteps.

Infinite-Horizon VI: Operator View

- $V \in \mathbb{R}^{|\mathcal{S}|}$
- VI update is a function $\mathcal{T} : \mathbb{R}^{|\mathcal{S}|} \to \mathbb{R}^{|\mathcal{S}|}$, called **backup operator**

$$[\mathcal{T}V](s) = \max_a \mathbb{E}_{s' \mid s,a} [r + \gamma V(s')]$$

Algorithm 3 Infinite-Horizon Value Iteration (v2)

- Initialize $V^{(0)}$ arbitrarily.
  for $n = 0, 1, 2, \ldots$ until termination condition **do**
  $V^{(n+1)} = \mathcal{T}V^{(n)}$
  end for

Contraction Mapping View

- Backup operator $\mathcal{T}$ is a contraction with modulus γ under ∞-norm

$$\|\mathcal{T}V - \mathcal{T}W\|_\infty \leq \gamma \|V - W\|_\infty$$

- By contraction-mapping principle, B has a fixed point, called V^*, and iterates $V, \mathcal{T}V, \mathcal{T}^2V, \cdots \to V^{*,\gamma}$.

Policy Evaluation

- Problem: how to evaluate fixed policy π:

$$V^{\pi,\gamma}(s) = \mathbb{E}\left[r_0 + \gamma r_1 + \gamma^2 r_2 + \ldots \mid s_0 = s\right]$$

- Can consider finite-horizon problem

$$\begin{aligned}&\mathbb{E}\left[r_0 + r_1 + \cdots + r_{T-1} + v_T(s_T)\right]\\&\quad = \mathbb{E}\left[r_0 + \gamma\mathbb{E}\left[r_1 + \cdots + \gamma\mathbb{E}\left[r_{T-1} + V_T(s_T)\right]\right]\right]\end{aligned}$$

- Backwards recursion involves a backup operation $V_t = \mathcal{T}^\pi V_{t+1}$, where

$$[\mathcal{T}^\pi V](s) = \mathbb{E}_{s' \mid s, a=\pi(s)}\left[r + \gamma V(s')\right]$$

- $\mathcal{T}^\pi$ is also a contraction with modulus γ, sequence $V, \mathcal{T}^\pi V, (\mathcal{T}^\pi)^2 V, \cdots \to V^{\pi,\gamma}$
- $V = \mathcal{T}^\pi V$ is a linear equation that we can solve exactly: $V(s) = \sum_{s'} P(s' \mid s, a = \pi(s))[r(s,a,s') + \gamma V(s')]$

Policy Iteration: Overview

- Alternate between
  1. Evaluate policy $\pi \Rightarrow V^\pi$
  2. Set new policy to be *greedy* policy for V^π

$$\pi(s) = \arg\max_a \mathbb{E}_{s' \mid s,a} \left[r + \gamma V^\pi(s') \right]$$

- Guaranteed to converge to optimal policy and value function in a finite number of iterations, when $\gamma < 1$
- Value function converges faster than in value iteration[1]

[1] M. L. Puterman. *Markov decision processes: discrete stochastic dynamic programming*. John Wiley & Sons, 2014.

Policy Iteration: Operator Form

Algorithm 4 Policy Iteration

Initialize $\pi^{(0)}$.

for $n = 1, 2, \ldots$ **do**

$V^{(n-1)} = \mathrm{Solve}[V = \mathcal{T}^{\pi^{(n-1)}} V]$

$\pi^{(n)} = \mathcal{G} V^{\pi^{(n-1)}}$

end for

Policy Iteration: Convergence

- Policy sequence $\pi^{(0)}, \pi^{(1)}, \pi^{(2)}, \ldots$ is monotonically improving, with nondecreasing value function: $V^{\pi^{(0)}} \leq V^{\pi^{(1)}} \leq V^{\pi^{(2)}} \leq \ldots$. Informal argument:
  - Switch policy at first timestep from $\pi^{(0)}$ to $\pi^{(1)}$
    - Before: $V(s_0) = \mathbb{E}_{s_1 \mid s_0, a_0 = \pi(s_0)} \left[r_0 + \gamma V^{\pi}(s_1) \right]$
    - After: $V(s_0) = \max_{a_0} \mathbb{E}_{s_1 \mid s_0, a_0 = \pi(s_0)} \left[r_0 + \gamma V(s_1) \right]$
  - $V_{\pi^{(1)}\pi^{(0)}\pi^{(0)}\pi^{(0)}\ldots} \geq V_{\pi^{(0)}\pi^{(0)}\pi^{(0)}\pi^{(0)}\ldots}$
  - $V_{\pi^{(1)}\pi^{(1)}\pi^{(0)}\pi^{(0)}\ldots} \geq V_{\pi^{(1)}\pi^{(0)}\pi^{(0)}\pi^{(0)}\ldots}$
  - $\ldots \Rightarrow$
    $V_{\pi^{(1)}\pi^{(1)}\pi^{(1)}\pi^{(1)}\ldots} \geq V_{\pi^{(0)}\pi^{(0)}\pi^{(0)}\pi^{(0)}\ldots}$
- If the value function does not increase, then we're done: $V_{\pi^{(n)}} = V_{\pi^{(n+1)}} \Rightarrow V_{\pi^{(n)}} = \mathcal{T} V_{\pi^{(n)}} \Rightarrow \pi^{(n)} = \pi^*$.

Modified Policy Iteration

- Update π to be the greedy policy, then value function with k backups (k-step lookahead)

Algorithm 5 Modified Policy Iteration

- Initialize $V^{(0)}$.

for $n = 1, 2, \ldots$ **do**

$\pi^{(n+1)} = GV^{(n)}$

$V^{(n+1)} = (\mathcal{T}^{\pi^{(n+1)}})^k V^{(n)}$, for integer $k \geq 1$

end for

- $k = 1$: value iteration
- $k = \infty$: policy iteration
- See Puterman's textbook[2] for more details

[2] M. L. Puterman. *Markov decision processes: discrete stochastic dynamic programming*. John Wiley & Sons, 2014.

The End

- Homework: will be released later today or early tomorrow, due on Feb 22
- Next time: policy gradient methods: infinitesimal policy iteration